X irred., proj. variety, dim n, L a line bundle on X.

The <u>volume</u> of L is $\mathrm{vol}(L) = \limsup_{m\to\infty} \dfrac{h^0(L^{\otimes m})}{m^n/n!}$

$(\mathrm{vol}(D) := \mathrm{vol}(\mathcal{O}(D)))$

So $\overset{\in \mathrm{Div}(X)}{D}$ is big $\iff h^0(L^{\otimes m})$ grows like $C\cdot m^n$ for some $C>0$
$\iff \mathrm{vol}(L) > 0$

<u>Rmk</u>: If D is ample, Asymp R-R and Serre Vanishing $\Rightarrow \mathrm{vol}(D) = D^n$.

<u>Claim</u>: If D is nef, $\mathrm{vol}(D) = D^n$.

<u>Pf</u>: Fix A ample.
Fujita's vanishing thm $\Rightarrow h^i(k\mathbf{A}+B) = 0$, for $k \geq k(H)$, $i>0$
and <u>any</u> nef B.

Thus, choosing k suff. large so that $H := kA$ is v. ample,

we have $h^i(H + mD) = 0$ for $m \geq 0$, $i > 0$.

<u>Exercise</u>: Use this + Asymptotic R-R to show
$$h^0(mD) = \frac{D^n}{n!} m^n + O(m^{n-1}).$$

Thus, by the exercise, $\mathrm{vol}(D) = D^n$. $\square$

Ex: Recall from an earlier proof that if D, E are nef, then

$$h^0(\mathcal{O}_X(m(D-E))) \geq \left(D^n - n(D^{n-1}\cdot E)\right)\frac{m^n}{n!} + O(m^{n-1})$$

$$\Rightarrow \operatorname{Vol}(D-E) \geq D^n - n\, D^{n-1}\cdot E.$$

Ex: If $X = \mathbb{P}^n$, $L = \mathcal{O}(d)$, $d \geq 0$, then

$$h^0(mL) = \binom{n+md}{n} = \frac{(md)^n}{n!} + O(m^{n-1})$$

$$\Rightarrow \operatorname{vol}(L) = d^n.$$

Lemma: L big, A very ample on X.
If $E, E' \in |A|$ are very general, then

$$\operatorname{Vol}_E(L|_E) = \operatorname{vol}_{E'}(L|_{E'})$$

Pf: For each $m > 0$, $h^0(L^{\otimes m}|_E)$ is semicontinuous on $|A|$.

So there is an open $U_m \subseteq |A|$ s.t. $h^0(L^{\otimes m}|_E) = h^0(L^{\otimes m}|_{E'})$

So we can choose $E, E' \subseteq \bigcap_{m>0} U_m$. $\square$

Lemma: D a divisor on X, $a > 0$ fixed integer.

Then $$\limsup_m \frac{h^0(\mathcal{O}(mD))}{m^n/n!} = \limsup_k \frac{h^0(\mathcal{O}_X(akD))}{(ak)^n/n!}$$

Pf: If D isn't big, both sides $= 0$. So assume D is big.

For each $r \in \mathbb{N}$, define

$$V_r = \limsup_k \frac{h^0((ak+r)D)}{(ak+r)^n/n!}$$

so we want: $V_0 = \mathrm{vol}(D)$

Note that V_r describes a subsequence of the sequence computing the volume, jumping by a indices each time.

Thus for any integer $r_0 \geq 0$, $\mathrm{vol}(D) = \max\{V_{r_0+1}, \dots, V_{r_0+a}\}$.

$\Rightarrow$ suffices to find r_0 s.t. $V_0 = V_r$ for $r \in [r_0+1, r_0+a]$

Since D is big, $e(D) = 1$, so we can find $r_0 > 0$ s.t.

$$h^0(rD) > 0 \quad \text{for } r \geq r_0.$$

Then $\mathcal{O} \subseteq \mathcal{O}(rD)$

$\Rightarrow \mathcal{O}(kaD) \subseteq \mathcal{O}((ka+r)D)$ for $r \geq r_0$, $k \in \mathbb{N}$

Choose $q > 0$ suff. large s.t. $qa - (r_0 + a) \geq r_0$

Then, similarly,

$$\mathcal{O} \subseteq \mathcal{O}((qa-r)D) \qquad \text{for} \quad r \leq r_0 + a$$

$$\Rightarrow \mathcal{O}((ka+r)D) \subseteq \mathcal{O}((ka+qa)D)$$

Thus, for $k \in \mathbb{N}$ and $r_0+1 \leq r \leq r_0 + a$

$$h^0(kaD) \leq h^0((ka+r)D) \leq h^0((k+q)aD) \qquad (*)$$

So
$$\overset{V_0}{\underset{k}{\limsup}} \frac{h^0(kaD)}{(ka)^n/n!} \leq \limsup \frac{h^0((ka+r)D)}{(ka)^n/n!} = \overset{V_r}{\limsup} \frac{h^0((ka+r)D)}{(ka+r)^n/n!}$$

$$\leq \limsup_k \frac{h^0((k+q)a\,D)}{(ka)^n/n!} = \limsup_k \frac{h^0((k+q)aD)}{((k+q)a)^n/n!} = V_0$$

(throw away first q terms)

$\Rightarrow V_0 = V_r$ for $r \in [r_0+1, r_0+a]$. $\square$

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Properties of volume:

Let $n = \dim X$, D a big divisor.

(i.) $\operatorname{vol}(aD) = a^n \operatorname{vol}(D)$ (holds for arb. D)

(ii.) If $N \in \operatorname{Div}(X)$, then

$$\lim_{p\to\infty}\left(\frac{1}{p^n}\operatorname{vol}(pD-N)\right) = \operatorname{vol}(D)$$

Pf: (i.)

$$\mathrm{vol}(aD) = \limsup_k \frac{h^0(akD)}{k^n/n!} = a^n \limsup_k \frac{h^0(akD)}{(ak)^n/n!} = a^n \mathrm{vol}(D)$$

by lemma from last time

(ii.) D big, N arbitrary div.

Want to reduce to N effective:

Find A, B effective s.t. $N \equiv_{lin} A - B$.

By Kodaira's lemma, $\exists\ r > 0$ s.t.

$$rD - B \equiv_{lin} B_1 \text{ effective.}$$

$$\Rightarrow B \equiv rD - B_1$$

$$\Rightarrow pD - N \equiv pD - A + (rD - B_1)$$

$$= (p+r)D - \underbrace{(A + B_1)}_{\text{effective}}.$$

$$\Rightarrow \lim_{p\to\infty} \frac{1}{p^n} \mathrm{vol}(pD - N) = \lim_{p\to\infty} \frac{1}{p^n} \mathrm{vol}((p+r)D - (A + B_1))$$

$$= \lim_{p\to\infty} \frac{1}{(p+r)^n} \mathrm{vol}((p+r)D - (A+B_1))$$

$$= \lim_{p\to\infty} \frac{1}{p^n} \mathrm{vol}(pD - (A+B_1))$$

Thus, replacing N w/ $A+B_1$, we can assume N is effective.

If N' is another effective divisor, then

$$\mathrm{vol}(pD - (N+N')) \leq \mathrm{vol}(pD - N) \leq \mathrm{vol}(pD).$$

$$\Rightarrow \lim_{p\to\infty} \frac{1}{p^n} \mathrm{vol}(pD - (N+N')) \leq \lim_{p\to\infty} \frac{1}{p^n} \mathrm{vol}(pD - N) \leq \lim_{p\to\infty} \frac{1}{p^n} p^n \mathrm{vol}\, D = \mathrm{vol}\, D$$

Thus, we can replace N with $N+N'$, any effective N'.

Specifically, choosing N' suff. large multiple of ample, can assume N v. ample.

Then just like in a previous argument, can choose $E_1, \ldots, E_m \in |N|$ v. general so that

$$0 \to H^0(\mathcal{O}_X(m(pD-N))) \to H^0(\mathcal{O}_X(mpD)) \to \bigoplus_{i=1}^{m} H^0(\mathcal{O}_{E_i}(mpD))$$

is exact, and for each m, $h^0(\mathcal{O}_{E_i}(mpD))$ is indep of i,

So just choosing a single $E \in |N|$ v. general,

$$h^0(\mathcal{O}_X(m(pD-N))) \geq h^0(\mathcal{O}_X(mpD)) - m \cdot h^0(\mathcal{O}_E(mpD))$$

$$\Rightarrow \mathrm{vol}_X(pD-N) \geq \mathrm{vol}_X(pD) - \underbrace{\limsup_m \frac{m \cdot h^0(\mathcal{O}_E(mpD))}{m^n/n!}}_{= n \cdot \mathrm{vol}_E(pD|_E)}$$

By (i.), $\mathrm{vol}_E(pD|_E) = p^{n-1}\,\mathrm{vol}_E(D|_E)$.

$$\Rightarrow \mathrm{vol}_X(pD-N) \geq \mathrm{vol}_X(pD) - O(p^{n-1})$$

$$\mathrm{vol}_X(D) \leq \lim_{p\to\infty} \frac{1}{p^n} \mathrm{vol}_X(pD-N) \leq \mathrm{vol}(D). \quad \square$$

<u>Rmk</u>: Equivalently, we could say

$$\lim_{p\to\infty}\left(\frac{1}{p^n}\left|\mathrm{vol}(pD-N)-\mathrm{vol}(pD)\right|\right)=0.$$

Volume of $\mathbb{Q}$-divisors

Def: If $D\in \mathrm{Div}_{\mathbb{Q}}(X)$, we define

$$\mathrm{vol}(D)=\limsup_{\substack{m:\, mD\\ \text{integral}}}\frac{h^0(mD)}{m^n/n!}$$

Rmk: If we take $a=$ l.c.m. of denominators of D,

we get
$$\mathrm{vol}(D)=\limsup_{k}\frac{h^0(kaD)}{(ka)^n D}=\frac{1}{\alpha^n}\mathrm{vol}(aD).$$

In fact, from previous lemma, if $b>0$ is an integer s.t. bD is integral,

then
$$\frac{1}{(ab)^n}\mathrm{vol}(abD)=\frac{1}{b^n}\mathrm{vol}(bD)=\frac{1}{a^n}\mathrm{vol}(aD)=\mathrm{vol}(D).$$

(Easier def. to work with).

Prop: If $D\equiv_{num}D'$ $(\in \mathrm{Div}_{\mathbb{R}}(X))$ then $\mathrm{vol}(D)=\mathrm{vol}(D')$.

Rmk: If D, D' are $\mathbb{Q}$-divisors s.t. mD and mD' are integral,

then $D\equiv_{num}D' \Rightarrow mD\equiv_{num}mD'$,

$$\Rightarrow \underset{\mathrm{vol}(D)}{\underset{\shortparallel}{\frac{1}{m^n}\mathrm{vol}(mD)}}=\underset{\mathrm{vol}(D')}{\underset{\shortparallel}{\frac{1}{m^n}\mathrm{vol}(mD')}}$$

First need some lemmas + definition:

Def: (C-M regularity) $\mathcal{F}$ a sheaf on X projective, B an ample, globally generated line bundle. $\mathcal{F}$ is <u>m-regular</u> w.r.t. B if

$$H^i(\mathcal{F} \otimes B^{\otimes(m-i)}) = 0 \text{ for } i > 0.$$

Exer: If $\mathcal{F}$ is m-regular w.r.t. B, then $\forall\ k \geq 0$, $\mathcal{F} \otimes B^{\otimes(m+k)}$ is globally generated.

Lemma: $\exists$ a fixed $N \in \mathrm{Div}(X)$ s.t.

$$H^0(\mathcal{O}_X(N+P)) \neq 0$$

for every $P \equiv_{num} 0$.

(In particular if $P_0 \equiv_{num} 0$, then for $m \in \mathbb{Z}$, $N + mP_0$ is lin. equivalent to an effective divisor.)

Pf: Let B be v. ample on X.

Fujita vanishing $\Rightarrow$ if $F = mB$ for m suff large, then

$$H^i(\mathcal{O}_X(F+A)) = 0 \quad \text{for } i > 0 \text{ and } \underline{\text{every}} \text{ nef } A.$$

$P + (n-i)B$ is nef for $i \leq n$, (where $P \equiv_{num} 0$) so

$$\Rightarrow H^i(\mathcal{O}_X(F + P + (n-i)B)) = 0 \quad \forall\ i > 0.$$

$\Rightarrow \mathcal{O}_X(F+P)$ is n-regular w.r.t. $\mathcal{O}_X(B)$

$\Rightarrow$ by exercise, $F + nB + P$ is globally generated

$\Rightarrow$ setting $N = F + nB$, $H^0(\mathcal{O}_X(N+P)) \neq 0$. $\square$

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Pf of Prop: If D isn't big, then $\mathrm{vol}(D) = 0 = \mathrm{vol}(D')$.

So assume D, D' are big.

Need to show $\mathrm{vol}(D) = \mathrm{vol}(D+P)$ for any $\overset{\mathrm{Div}(X)}{\overset{\cup\!\!\!\!\!\shortmid}{P}} \equiv_{\mathrm{num}} 0$.

Take N from lemma. Then for any integer $p > 0$,

$$H^0(\mathcal{O}_X(N - pP)) \neq 0.$$

$\Rightarrow$ can choose $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(m(N - pP))$ for every $m \geq 0$

$$\Rightarrow \underbrace{\mathcal{O}_X(mpD - m(N - pP))}_{\mathcal{O}_X(m(p(D+P)-N))} \hookrightarrow \mathcal{O}_X(mpD)$$

Thus, $\mathrm{vol}(p(D+P) - N) \leq \mathrm{vol}(pD) = p^n \mathrm{vol}(D)$.

But $D+P$ is big, so by "Property of volume (ii)",

$$\lim_{p \to \infty} \frac{1}{p^n} \mathrm{vol}(p(D+P) - N) = \mathrm{vol}(D+P)$$

$\Rightarrow \mathrm{vol}(D+P) \leq \mathrm{vol}(D)$.

Similarly, $\mathrm{vol}((D+P) - P) \leq \mathrm{vol}(D+P)$. $\square$

Volume is also a birational invariant:

<u>Prop</u>: Let $\nu : X' \to X$ be a projective birational morphism, X', X irreducible, dim n. Let $D \in \mathrm{Div}_{\mathbb{Q}}(X)$, and $D' = \nu^* D$. Then

$$\mathrm{vol}_{X'}(D') = \mathrm{vol}_X(D).$$

<u>Pf</u>: Choose $m > 0$ s.t. mD and mD' are integral. Then

$$\mathrm{vol}(D') = \mathrm{vol}(D) \Leftrightarrow \mathrm{vol}(mD') = \mathrm{vol}(mD).$$

Thus, assume D and D' are integral.

Projection formula $\Rightarrow$ $H^0(\mathcal{O}_{X'}(mD')) = H^0(\nu_* \mathcal{O}_{X'} \otimes \mathcal{O}_X(mD))$

Just as in a previous argument, $\quad 0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_{X'} \to \eta \to 0$

where η is supported in dim $\leq n-1$.

$$\Rightarrow h^0(\mathcal{O}_X(mD)) \leq h^0(\mathcal{O}_{X'}(mD'))$$
$$\leq h^0(\mathcal{O}_X(mD)) + \underbrace{h^0(\eta(mD))}_{O(m^{n-1})}$$

$\Rightarrow \mathrm{vol}(D) = \mathrm{vol}(D')$. $\square$

Continuity of volume:

<u>Thm</u>: X irr., projective, dim n.

Let $\| \ \|$ be a fixed norm on $N^1(X)_{\mathbb{R}}$ inducing standard topology. Then there is a constant $C>0$ s.t. for any $\gamma, \gamma' \in N^1(X)_{\mathbb{Q}}$

$$\left| \mathrm{vol}(\gamma) - \mathrm{vol}(\gamma') \right| \le C \cdot \|\gamma - \gamma'\| \cdot \left(\max(\|\gamma\|, \|\gamma'\|)\right)^{n-1}.$$

Pf: Both sides of inequality are homogeneous in (γ, γ') (of deg n). i.e. the inequality is invariant under replacing γ and γ' by the same multiple.

Choose $A_1, \dots, A_r$ a basis of very ample divisors in $N^1(X)_{\mathbb{Q}}$

Then we can prove the inequality for divisors

$$D = a_1 \cdot A_1 + \dots + a_r A_r$$
$$D' = a_1' A_1 + \dots + a_r' A_r$$

Where $a_i, a_i' \in \mathbb{Z}$ (i.e. by replacing γ, and γ' by suff. multiples)

Choose the norm to be $\|D\| = \max\{|a_i|\}$.

First assume $a_i - a_i' = b_i \ge 0$.

Then $D' = D - B$ where $B = \sum b_i \cdot A_i$, effective

Take $E_j \in |A_j|$ very general.

Claim: $\mathrm{vol}(D-B) \ge \mathrm{vol}(D) - n \sum b_j \cdot \mathrm{vol}_{E_j}(D_j)$ $\quad (*)$

where $D_j = D|_{E_j}$.

The claim follows easily by induction on $\#\{i \mid b_i \neq 0\}$, so we just need to prove the base case: $B = b_1 A_1$. (WLOG)

Take $m \gg 0$, and choose mb_1 v. general diviors $F_\alpha \in |A_1|$.

Have exact sequence: $0 \longrightarrow h^0(\mathcal{O}_X(mD - mb_1 A_1)) \longrightarrow h^0(\mathcal{O}_X(mD)) \longrightarrow \bigoplus_{\alpha=1}^{mb_1} h^0(\mathcal{O}_{F_\alpha}(mD))$

Just as in a previous proof, since F_α were very general, the claim follows.

Now let $D^+ = \sum |a_i| A_i$ and $D_j^+ = D^+|_{E_j}$.

Then $D_j^+ - D_j$ is effective, so $\mathcal{O}(D_j) \subseteq \mathcal{O}(D_j^+)$

$\Rightarrow \mathrm{vol}_{E_j}(D_j) \leq \mathrm{vol}_{E_j}(D_j^+)$

D^+ is a $\geq$ lin. comb. of ample divisors, so D_j^+ is nef.

$\Rightarrow \mathrm{vol}_{E_j}(D_j^+) = (D_j^+)^{n-1} \cdot E_j = \left(\sum_i |a_i| \cdot A_i\right)^{n-1} \cdot A_j$

$\leq (\max\{|a_i|\})^{n-1} \cdot C_j = C_j (\|D\|)^{n-1}$

for some constant $C_j > 0$ (depending on j and the A_i's)

Then $(*)$ implies

$$\mathrm{vol}(D) - \mathrm{vol}(\overset{D'}{\overset{\shortparallel}{D-B}}) \le n \cdot \sum \left(b_j \cdot \mathrm{vol}_{E_j}(D_j)\right)$$

$$\le n \max\{b_i\} \sum_j C_j (\|D\|)^{n-1}$$

$$\le \|B\| \cdot \underbrace{n \cdot \max\{C_j\} \cdot r}_{\overset{\shortparallel}{C}} \cdot \|D\|^{n-1} = C \cdot \|D\|^{n-1} \cdot \|D - D'\| \qquad (**)$$

as desired.

More generally, assume $D' = D + E - F$, where E, F effective, integral, w/ no nonzero A_i-"coordinates" in common.

Then $0 \le \mathrm{vol}(D) - \mathrm{vol}(D - F) \le C (\|D\|)^{n-1} \cdot \|F\|$ by $(**)$

and $0 \le \mathrm{vol}(D+E-F) - \mathrm{vol}((D+E-F)-E) \le C(\|D+E-F\|)^{n-1} \cdot \|E\|$

$$\Rightarrow \left|\mathrm{vol}(D) - \mathrm{vol}(\underbrace{D+E-F}_{D'})\right| \le C \left(\max(\|D\|, \|D'\|)\right)^{n-1} \cdot \underbrace{\max(\|E\|, \|F\|)}_{\substack{\|E - F\| \\ \shortparallel \\ \|D' - D\|}} \qquad \square$$

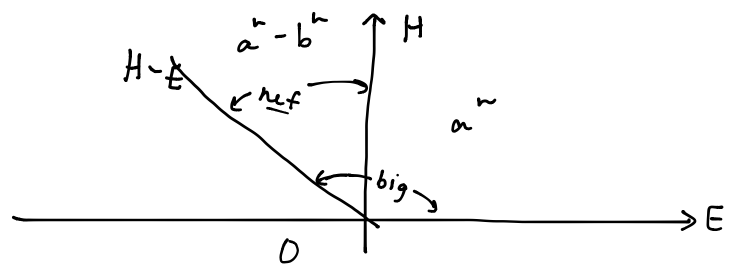
<u>Cor</u>: The volume function on $N^1(X)_{\mathbb{Q}}$ extends uniquely to a continuous function on $N^1(X)_{\mathbb{R}}$.

(Take $\underset{\substack{\cap \\ N^1(X)_{\mathbb{Q}}}}{z_i} \to z \in N^1(X)_{\mathbb{R}}$. Then $\lim_{i \to \infty} \mathrm{vol}(z_i)$ exists and is indep of $\{z_i\}$.)

Apr 12

<u>Ex</u>: $X = Bl_p(\mathbb{P}^n)$. $H, E \in N_1(X)_{\mathbb{R}} = \mathbb{R}^2$

H: pullback of hyperplane; E: exceptional.



The nef cone is bounded by H and $H-E$.

So if $D \equiv aH - bE$ where $0 \le b \le a$, then

$$\mathrm{vol}(D) = D^n = a^n + b^n(-E)^n = a^n - b^n$$

If $D \equiv aH - bE$, $a, -b \ge 0$

then $-bE$ is a fixed component. i.e. $|aH - bE| \cong |aH|$

$$\Rightarrow \mathrm{vol}(D) = \mathrm{vol}(aH) = a^n$$

Many of the statements about vol for $\mathbb{Z}$-divisors now follow for $\mathbb{R}$-divisors by continuity of volume:

<u>Cor</u>: X projective, irreducible, dim n.

(i.) If $\xi, \eta \in \mathrm{Nef}(X)_{\mathbb{R}}$, then $\mathrm{vol}(\xi - \eta) \ge (\xi^n) - n(\xi^{n-1} \cdot \eta)$

(ii.) If $\gamma, e \in N^1(X)_{\mathbb{R}}$ are big and effective, resp. then

$$\mathrm{vol}(\gamma) \le \mathrm{vol}(\gamma + e)$$

(iii.) $\nu : X' \to X$ birational (X' irred, projective), $\alpha \in N^1(X)_{\mathbb{R}}$

then $\mathrm{vol}_X(\alpha) = \mathrm{vol}_{X'}(\nu^* \alpha)$